FILTERING FOR LINEAR DISTRIBUTED PARAMETER SYSTEMS

by

Harold J. Kushner[*]

Center for Dynamical Systems
Brown University
Providence, Rhode Island

FILTERING FOR LINEAR DISTRIBUTED PARAMETER SYSTEMS

Harold J. Kushner

1. Introduction

The systems to be considered are described by parabolic equations
with 'white noise' inputs. We are interested in conditions which
guarantee that the solution $U(x,t)$, a random surface, has certain
smoothness properties, and also in the smoothness properties of the con-
ditional expectation $E[U(x,t)|$ given data up to $t]$. Such results are
developed in [1], [2] using the Sobolev imbedding theorem.

First, some of these results will be stated. A system model (first
boundary value problem) is discussed in Section 2, Lemma 3. The noisy
observations for this problem have the form (7). Lemma 4 proves the
smoothness of the conditional mean and covariance, and Theorem 1 gives
the form of the optimal filter. Section 3 considers a second boundary
value problem (16) with surface observations of the form (B6). Lemma 5
proves the smoothness of the solution to (16), and Theorem 2 gives the
form of the optimal filter.

Smoothness Results on Random Surfaces. Let z_t be a normalized
Wiener process, D a bounded open domain in E^n with closure $\bar{D}$ and
a continuous and piecewise uniformly differentiable boundary and write
$\bar{R} = \bar{D} \times [0,T]$. Let $D_t = \partial/\partial t$, $D_i = \partial/\partial x_i$, $D_i^\ell = \partial^\ell/\partial x_j^\ell$. Let $f(x,t)$
be a stochastic process on $\bar{D} \times [0,T] = \bar{R}$. The parenthesis in
$(D_i f(x,t))$, denotes the 'mean square' derivative of $f(x,t)$ with respect

to x_i, if it exists. Define the norm

$$\|g(x)\|_{W_{\ell,p}(\overline{D})} = \sum_{k=0}^{\ell} \sum_{\ell_1+\cdots+\ell_n=k} \|D_1^{\ell_1}\cdots D_n^{\ell_n} g(x)\|_{L_p(\overline{D})}. \qquad (1)$$

where $\psi \in L_p(\overline{D})$ means that $\int_{\overline{D}} |\psi(x)|^p dx \equiv \|\psi\|_{L_p(\overline{D})}^p < \infty$. References

[1] and [2], from which Lemmas 1 and 2 are taken give conditions on the

expectations of integrals of powers of the 'mean square' derivatives,

which guarantee that $f(x,t)$ has a w.p.1. continuous version on $\overline{R}$, and

perhaps several continuous derivatives with respect to components of x.

The proof of Lemma 1 is contained in [2].

Lemma 1. Let the boundary ∂D of D have the property that any

line intersects it only finitely often. Let the functions

$$\alpha(x,t,s), \; \{D_i\alpha(x,t,s)\}, \; \{D_iD_j\alpha(x,t,s)\},$$

$$\{D_iD_jD_k\alpha(x,t,s)\}, \; \{D_iD_jD_kD_\ell\alpha(x,t,s)\} \qquad (*)$$

be defined on $\overline{D} \times [0,T] \times [0,T] = \overline{R} \times [0,T]$, continuous in (x,t) for

each s, and bounded (in absolute value) by a square integrable function

of s. Let f be any function in the set $(*)$, and let z(t) be a

Wiener process. Then $\int_t^T f^2(x,t,s)ds \leq M < \infty$ for some real number M,

and $\int_0^t f(x,t,s)dz_s$ can be defined to be a separable and measurable

process with parameter (x,t). There is a null set N and a separable

and measurable version of $\int_0^t \alpha(x,t,s)dz_s = \psi(x,t)$ which, for $\omega \notin N$, is

continuous in (x,t) and has three continuous (in (x,t)) derivatives

with respect to the components of x. These derivatives are equal to continuous (for $\omega \notin N$), separable and measurable versions of
$\int_0^t D_i \alpha(x,t,s)dz_s$, $\int_0^t D_i D_j \alpha(x,t,s)dz_s$, $\int_0^t D_i D_j D_k \alpha(x,t,s)dz_s$, respectively.

Let in addition, for some real numbers $K < \infty$, $\beta > 0$,

$$E\left\{\int_0^{t+\Delta} f(x,t+\Delta,s)dz_s - \int_0^t f(x,t,s)dz_s\right\}^2 \qquad (**)$$

$$= \int_0^t [f(x,t+\Delta,s) - f(x,t,s)]^2 ds + \int_t^{t+\Delta} f^2(x,t+\Delta,s)ds \le K\Delta^\beta,$$

where f is any member of $(*)$. Let g be any member of the first three sets of $(*)$. Then the continuous version (for $\omega \notin N$) of
$\int_0^t g(x,t,s)dz_s = \phi(x,t)$ is Holder continuous on $\bar{R}$, i.e., there is some $K(\omega) < \infty$ w.p.l. and a real $\gamma > 0$ so that

$$|\phi(x+\delta,t+\Delta) - \phi(x,t)| \le K(\omega)[|\Delta|^\gamma + |\delta|^\gamma],$$

where $|\cdot|$ refers to the Euclidean norm.

Lemma 2. Let $f(x,t)$ be a process on $\bar{R}$, which is continuous in probability together with its 'mean square' derivatives up to order ℓ on $\bar{R}$. Let $p\ell > n$, $p > 1$, and suppose that[†] for $0 \le s \le t \le T$,

---

[†]Recall that (2) is equivalent to
$$E[\int_D |(D_1^{\ell_1} \cdots D_n^{\ell_n}\{f(x,t) - f(x,s)\})|^p dx]^{q/p} \le K|t-s|^{1+\alpha}$$
for all $\ell_1 + \cdots + \ell_n \le k \le \ell$, for $0 \le s \le t \le T$.

$$E\|f(\cdot,t) - f(\cdot,s)\|_{W_{\ell,p}(\overline{D})}^{q} \leq K|t-s|^{1+\alpha} \tag{2}$$

for some real $K < \infty$ and $1 \leq q < \infty$ and $\alpha > 0$. Then there is a w.p.l. continuous version of $f(\cdot,\cdot)$ on $\overline{R} \times [0,T]$, and the version is Holder continuous in t, uniformly in x, w.p.l.

If $0 < m < \ell - n/p$, then the 'mean square' derivatives of order $\leq m$ have continuous versions on $\overline{R}$ w.p.l., and $f(x,t)$ has w.p.l. a continuous version whose first m x-derivatives coincide with the 'mean square' derivatives.

For proof, see Theorem 4 in [1].

2. Filtering for a Stochastic First Boundary Value Problem

System Model. The first system with which we will deal has the representation[†]

$$dU(x,t) = [\mathcal{L}U(x,t) + \int k(y,x,t)U(y,t)dy]dt + \sigma(x,t)dz, \qquad (4)$$

where

$$\mathcal{L} = \sum a_{ij}(x,t)D_iD_j + \sum b_i(x,t)D_i \qquad (5)$$

and (A1) - (A7) hold.

(A1) ∂D (the boundary of D) has a local representation with holder continuous 4th derivatives.

(A2) The coefficients of $\mathcal{L}$, and their first two derivatives are Holder continuous in $\overline{R}$.

(A3) $\sum a_{ij}\xi_i\xi_j \geqq K \sum \xi_i^2$ for some real $\infty > K > 0$.

(A4) σ and its first four x-derivatives are Holder continuous on $\overline{R}$.

(A5) σ and $\mathcal{L}\sigma$ go to zero as $x \rightarrow \partial D$.

(A6) $k(y,x,t)$ is bounded, measurable and Holder continuous in x,t, uniformly in y, and $k(y,x,t) \rightarrow 0$ as $x \rightarrow \partial D$.

(A7) $U(x,0)$ is Gaussian for each x, has a bounded variance, Holder

---

[†]For notational simplicity, we let the 'driving term' be $\sigma(x,t)dz$. It could be $\sum \sigma_i(x,t)dz_i$, where the z_i are independent. See Lemma 2.2, [2].

continuous second derivatives, and $U(x,0)$ and $\mathcal{L}U(x,0) \to 0$ as $x \to \partial D$. $U(x,0)$ is independent of z_t and of w_t (to be introduced below).

In [2], Lemmas 1 and 2 are applied to (4) to give it a precise definition and

Lemma 3.[†] (See [2], Lemma 3.2 for proof.) <u>Assume (A1) - (A7). Then there is a random function $U(x,t)$ on $(0,T] \times \overline{D}$ so that a version (for $\omega \notin N$, a null set) of the uniformly (in $(0,T] \times D$) 'mean square' continuous functions</u>

$$U(x,t), \; (D_i U(x,t)), \ldots, \; (D_i D_j D_k U(x,t)) \tag{6}$$

<u>are continuous on $(0,T] \times \overline{D}$ w.p.1.; these versions of the 'mean square' derivatives are true derivatives.</u> $U(x,t)$ <u>and $\mathcal{L}U(x,t) \to 0$ as $x \to \partial D$ (for $\omega \notin N$), $U(x,t) \to U(x,0)$ (for $\omega \notin N$, and uniformly in x) as $t \to 0$. The first three sets of (6) are Holder continuous in t, for $\omega \notin N$.</u> $U(x,t)$ <u>is a Markov process (with values in a state space of functions with Holder continuous second derivatives). The members of (6) are Gaussian, and have uniformly bounded variances. The variances of $U(x,t)$ and of $\mathcal{L}U(x,t)$ tend to zero as $x \to \partial D$.</u> $U(x,t)$ <u>is non-anticipative with respect to the z_t process and the Ito differential of $U(x,t)$ satisfies</u> (4). $U(x,t)$ <u>satisfies the condition (2) of Lemma 2, for $m = 3$, $\ell = 4$, and all large p, and some finite q and $\alpha > 0$. $(D_i D_j D_k D_\ell U(x,t))$ is also uniformly 'mean square' continuous in $(0,T] \times R$.</u>

[†] The smoothness in (A1), (A2), (A4) gives a $U(x,t)$ with continuous third x-derivatives, hence Holder continuous second derivatives. In the control problem in [2], we wanted $U(x,t)$ to have Holder continuous second derivatives. If only continuous second derivatives are required, then the differentiability requirements in (A1), (A2), (A4) can be reduced by 1.

The Filtering Problem. Let $\tilde{w}_t$ be a normalized Wiener process independent of the z_t process, and suppose that

(A8) $H(x,t)$ is a vector-valued function which is defined and continuous on $\overline{R}$.

(A9) $B(t)$ is continuous on $[0,T]$ and $B(t)B'(t) = \sum_t$ is strictly positive definite on $[0,T]$.

Define $w_s = \int_0^s B(\tau)d\tilde{w}_\tau = \int_0^s \sum_\tau^{1/2} d\tilde{w}_\tau$. Suppose that the data

$$y(s) = \int_0^s [\int H(x,\tau)U(x,\tau)dx]d\tau + \int_0^s B(\tau)d\tilde{w}_\tau \equiv \int_0^s h_\tau d\tau + w_s, \quad s \leq t \quad (7)$$

is available at time t. All introduced σ-algebras are assumed to be complete with respect to whatever measures are imposed on them; let[t] $\mathcal{F}_t$ be the minimal σ-algebra determined by $y(s)$, $s \leq t$. Let μ_1 be the measure determined by the processes $U(x,s)$, $s \leq t$ and $dy(s) = h_s ds + dw_s$, $s \leq t$, and μ_0 the measure determined by the processes $U(x,t)$ and $dy(s) = dw_s$, $s \leq t$. Let E_i^t denote the expectation with respect to μ_i, and conditioned on $\mathcal{F}_t$.

Define

$$M(x,t) = E_1^t U(x,t)$$
$$P(x,y,t) = E_1^t (U(x,t) - M(x,t))(U(y,t) - M(y,t))$$
$$= E_1 (U(x,t) - M(x,t))(U(y,t) - M(y,t)).$$

---

[t] To be more precise, let Ω be a function space with generic element $\omega = (\omega',\omega'')$, where ω' is a member of the space of bounded functions on $\overline{R}$, and ω'' is a member of the space of bounded functions on $[0,T]$, with values in the Euclidean m-dimensional space E^m, where m is the dimension of w_t and y_t. The terminology is used later. See part 1^0 of the proof of Theorem 1.

Let $\mathscr{L}_x P(x,y,t)$ denote $\mathscr{L}$ operating on $P(x,y,t)$ as a function of x. Lemma 4 proves that there is a version of the estimate $M(x,t)$ which is, w.p.1., as smooth as the signal $U(x,t)$.

Lemma 4. Assume (A1) - (A9). Then (excluding a null set independent of (x,t)) there are on $(0,T] \times \overline{D}$ continuous versions of the first four sets of the continuous in quadratic mean functions

$$M(x,t), (D_i M(x,t)), (D_i D_j M(x,t)), (D_i D_j D_k M(x,t))(D_i D_j D_k D_\ell M(x,t)); \quad (8)$$

also $M(x,t) \to M(x,0)$ as $t \to 0$ (and also in quadratic mean) and the first three sets of 'mean square' derivatives are true derivatives and $E_1^t \mathscr{L} U(x,t) = \mathscr{L} M(x,t)$; also $M(x,t)$ and $\mathscr{L} M(x,t) \to 0$ as $x \to \partial D$, and the first three sets of (7) have Holder continuous versions. $P(x,y,t)$ has continuous third derivatives in the components of x and y on $(0,T] \times D$, and $P(x,y,t)$ and $\mathscr{L}_x P(x,y,t)$ and $\mathscr{L}_y P(x,y,t) \to 0$ as $x \to \partial D$ or $y \to \partial D$. $P(x,y,t) \to P(x,y,0)$ as $t \to 0$.

Proof. $M(x,t)$ and the $(D_i M(x,t)), \ldots, (D_i D_j D_k D_\ell M(x,t))$ exist and are 'mean square' continuous in x-uniformly in (x,t) in $(0,T] \times D$. Also $E_1^t (D_i U(x,t)) = (D_i M(x,t))$ w.p.1. (as well as for the next three derivatives) for each x,t in $(0,T] \times D$. These assertions easily follow from estimates of the following type: let e_i be the ith coordinate direction in E^n, where n is the dimension of x. Then

$$E_1 \left| \frac{M(x+e_i \Delta, t) - M(x,t)}{\Delta} - E_1^t (D_i U(x,t)) \right|^2$$

$$= E_1 \left| E_1^t \left\{ \frac{U(x+e_i\Delta, t) - U(x,t)}{\Delta} - (D_i U(x,t)) \right\} \right|^2$$

$$\leq E_1 \left| \frac{U(x+e_i\Delta, t) - U(x,t)}{\Delta} - (D_i U(x,t)) \right|^2 \to 0 \qquad (*)$$

as $\Delta \to 0$, uniformly for x, t in $D \times (0, T]$.

Furthermore, $M(x,t)$ also satisfies the estimates (**) and (***)

$$E_1 |M(x,t)|^u = E_1 |E_1^t U(x,t)|^u \leq E_1 |U(x,t)|^u$$

$$E_1 |(D_i D_j D_k D_\ell M(x,t))|^u \leq E_1 |(D_i D_j D_k D_\ell U(x,t))|^u \qquad (**)$$

and

$$E_1 |M(x,t) - M(x,s)|^u = E_1 |E_1^t U(x,t) - E_1^s U(x,s)|^u \qquad (***)$$

$$\leq K E_1 |E_1^t (U(x,t) - U(x,s))|^u + K E_1 |E_1^t U(x,t) - E_1^s U(x,t)|^u$$

$$\leq K E_1 |U(x,t) - U(x,s)|^u + \epsilon_u(x,t,s).$$

For $u = 2$, $\epsilon_u(x,t,s) \leq K_1 |t-s|$ (hence for $u = 2r$, $\epsilon_u(x,t,s) \leq K_r |t-s|^r$). The inequality on $\epsilon_u(x,t,s)$ follows from the Gaussianness, and the uniform positive definiteness of Σ_s, which gives an upper limit on the rate at which information can be collected.

Estimates (*), (**), (***) imply that the $M(x,t), \ldots, (D_i D_j D_k D_\ell M(x,t))$ are uniformly 'mean square' continuous in $(0,T] \times D$.

The statements concerning the continuity of $M(x,t)$ and its derivatives

then follow from Lemmas 2 and 3 since by the estimates (**), (***) (and obvious similar estimates for $(D_i M(x,t)),\dots,(D_i D_j D_k D_\ell M(x,t))$, if $U(x,t)$ satisfies Lemma 2 for $m = 3$, so does $M(x,t)$. Since (***) is valid for $s = 0$, $M(x,t) \to M(x,0)$.

$M(x,t)$ and $\mathscr{L}M(x,t) \to 0$ as $x \to \partial D$ since both $U(x,t)$ and $\mathscr{L}U(x,t)$ and their variances $\to 0$ as $x \to \partial D$.

The asserted smoothness of $P(x,y,t)$ and its boundary properties follow from the continuity in quadratic mean of the elements of (8), (8'), $\mathscr{L}M(x,t)$ and $\mathscr{L}U(x,t)$.

$$U(x,t),\dots,(D_i D_j D_k D_\ell U(x,t)) \tag{8'}$$

(see, for example, Loeve [6], Sec. 34.2 for the type of calculations which are required.) Q.E.D.

Theorem 1. Assume (A1) - (A9). Then there is a version of $M(x,t)$ which has the Itô differential w.p.1.

$$dM(x,t) = [\mathscr{L}M(x,t) + \int k(\xi,x,t)M(\xi,t)d\xi]dt +$$
$$[dy - \int H(\xi,t)M(\xi,t)d\xi]'\Sigma_t^{-1}[\int H(\xi,t)P(\xi,x,t)d\xi] \tag{9}$$

and, for this version, w.p.1., $M(x,t)$ and $\mathscr{L}M(x,t) \to 0$ as $x \to \partial D$. Furthermore, $P(x,y,t)$ satisfies

$$P_t(x,y,t) = [\mathscr{L}_x + \mathscr{L}_y]P(x,y,t)$$
$$+ \int k(y,\xi,t)P(x,\xi,t)d\xi + \int k(\xi,x,t)P(\xi,y,t)d\xi \tag{10}$$
$$+ \sigma(x,t)\sigma(y,t) - [\int H(\xi,t)P(x,\xi,t)d\xi]'\Sigma_t^{-1}[\int H(\xi,t)P(\xi,y,t)d\xi].$$

$P(x,y,t)$, $\mathcal{L}_x P(x,y,t)$ and $\mathcal{L}_y P(x,y,t) \to 0$ as $x \to \partial D$.

Proof. For the sake of keeping a framework which will allow a generalization (not proved here) to non-linear systems, we take a slightly more general approach than necessary. The non-linear problem for ordinary stochastic Ito equations was treated in [3], however here, we follow a slightly different approach, due to Zakai [4], which gives the result under weaker conditions than those required in [3].

1°. μ_o and μ_1 are absolutely continuous with respect to one another and $d\mu_1/d\mu_o = \exp R_t$

$$R_t = -\frac{1}{2} \int_o^t h_s' \Sigma_s^{-1} h_s \, ds + \int_o^t h_s' \Sigma_s^{-1} dy_s.$$

Next, following Zakai [4], note that if $E_1 |f(\omega,t)| < \infty$, then (see Loeve, [5], Sec. 24.4)

$$E_1^t f(\omega,t) = \frac{E_o^t f(\omega,t)(d\mu_1/d\mu_o)}{E_o^t(d\mu_1/d\mu_o)} = \frac{E_o^t f(\omega,t)\exp R_t}{E_o^t \exp R_t}. \qquad (11)$$

2°. Write

$$F_t = [\mathcal{L}U(x,t) + \int k(y,x,t)U(y,t)dy].$$

Then, w.p.1., by virtue of Lemma 4,

$$E_1^t F_t = \mathcal{L}M(x,t) + \int k(y,x,t)M(y,t)dy.$$

In (11), let $f(\omega,t) = U(x,t)$. Both $U(x,t)$ and $\exp R_t$ are stochastic integrals and

$$d[\exp R_s] = (\exp R_s)h'_s\Sigma_s^{-1} \, dy_s.$$

Then Ito's lemma, applied to (11) yields

$$E_1^t U(x,t) = M(x,t) =$$

$$\frac{E_0^t[U(x,0) + \int_0^t \exp R_s(F_s ds + \sigma(x,s)dz_s)][\int_0^t U(x,s)(\exp R_s)h'_s\Sigma_s^{-1})dy_s]}{E_0^t[1 + \int_0^t (\exp R_s)h'_s\Sigma_s^{-1} \, dy_s]} \tag{13}$$

$$\equiv \frac{A_t}{B_t},$$

$$A_t = E_0^t \int_0^t dU(x,s)\exp R_s ds + E_0^t U(x,0)$$

$$= E_0^t \int_0^t [U(x,s)(\exp R_s)h'_s\Sigma_s^{-1} dy_s + (\exp R_s)(F_s ds + \sigma(x,s)dz_s)] + E_0^t U(x,0)$$

and where $dydz = 0$ is used to eliminate the $(dU(x,s))(d \exp R_s)$ term from A_t.

As in Kushner [3] or Zakai [4], it can be shown below that,[†] w.p.1.

[†]The demonstration of (14), by the method of [3] requires more stringent conditions on R_s and U, then by the method of [4].

The method of [4] is applicable under the conditions of the hypothesis of Theorem 1.

The method of [3] may also be applied, by applying it to a suitable sequence of bounded $F_s^\epsilon, h_s^\epsilon$ which converges to F_s and h_s in probability.

$$E_o^t \int_o^t (\exp R_s)[F_s ds + \sigma(x,s)dz_s] = \int_o^t [E_o^s (\exp R_s)F_s]ds \qquad (14)$$

$$E_o^t \int_o^t U(x,s)[(\exp R_s)h_s' \Sigma_s^{-1}]dy_s = \int_o^t [E_o^s U(x,s)(\exp R_s)h_s' \Sigma_s^{-1}]dy_s$$

$$E_o^t \int_o^t (\exp R_s)h_s' \Sigma_s^{-1} dy_s = \int_o^t [E_o^s (\exp R_s)h_s' \Sigma_s^{-1}]dy_s$$

where the second integrals are well-defined w.p.1. Assuming (14) now, we proceed exactly as in [3] and get

$$dM(x,t) = \frac{dA_t}{B_t} - \frac{A_t dB_t}{B_t^2} + \frac{A_t (dB_t)^2}{B_t^3} - \frac{(dA_t)(dB_t)}{B_t^2} \qquad (15)$$

where

$$(dB_t)^2 = E_o^t[(\exp R_t)h_t']\Sigma_t^{-1}[E_o^t(\exp R_t)h_t],$$

$$(dA_t)(dB_t) = [E_o^t U(x,t)(\exp R_t)h_t']\Sigma_t^{-1}[E_o^t (\exp R_t)h_t].$$

(9) is obtained by substituting in (12) and using the fact ((11)) that $E_o^t f = [E_o^t f \exp R_t]/E_o^t \exp R_t$.

3°. Similarly, dP is calculated from the expression

$$dP(x,y,t) = dE_1^t U(x,t)U(y,t) - dM(x,t)M(y,t).$$

To get $dE_1^t U(x,t)U(y,t)$, repeat the procedure starting with (11), where we now let $f(\omega,t) = U(x,t)U(y,t)$, and use the w.p.1. equalities

$$E_1^t U(x,t) \mathcal{L}_y U(y,t) = M(x,t) \mathcal{L}_y M(y,t)$$

$$+ E_1^t (U(x,t) - M(x,t)) \mathcal{L}_y (U(y,t) - M(y,t))$$

$$= M(x,t) \mathcal{L}_y M(y,t) + \mathcal{L}_y P(x,y,t).$$

The details are straightforward and are omitted.   Q.E.D.

3. The Second Boundary Value Problem

Now, we consider the equation

$$dU(x,t) = [\mathcal{L}U(x,t) - f(x,t)]dt - \sigma(x,t)dz, \qquad (16a)$$

$$U_V(x,t) + \beta(x,t)U(x,t) = g(x,t) + v(x,t)r(t)$$

$$\mathcal{L} = \sum a_{ij}(x,t)D_iD_j + \sum b_i(x,t)D_i \qquad (16b)$$

where $U_V(x,t)$ is the co-normal derivative[†] $\partial U/\partial V = \lim\limits_{\substack{y\to x \\ y\in D}} \dfrac{\partial U(y,t)}{\partial V(x)}$ at

x on ∂D, and (B1) - (B8) are assumed.

(B1) $\sum a_{ij}(x,t)\xi_i\xi_j \geq K \sum \xi_i^2$ for some real $K > 0$.

(B2) $a_{ij}(x,t)$ and $b_i(x,t)$ are Holder continuous in R.

(B3) $f(x,t)$ is continuous, and Holder continuous in x, uniformly in t.

(B4) ∂D has a local representation with Holder continuous derivatives.

(B5) Real-valued $g(x,t)$ and row-vector valued $v(x,t)$ are continuous on $\overline{R}$ and r is the Gaussian random process satisfying $dr = A(t)rdt + G(t)d\tilde{z}$, where $\tilde{z}_t$ is independent the z_t and w_t processes introduced earlier, and of $U(x,0)$. $A(t)$ and $G(t)$ are bounded continuous functions.

(B6) The observations $dy = [\int_{\partial D} H(\xi,t)U(\xi,t)dS_\xi]dt + dw$ are taken, where $H(\xi,t)$ is continuous on $\partial D \times [0,T]$, and w_t is independent of $U(x,0)$, and dS_ξ is the differential surface measure on ∂D. Also Σ_t satisfies (A9), where $dw = \Sigma_t^{1/2}d\tilde{w}$, and $\tilde{w}_t$ is a normalized Wiener process.

[†]$V(x)$ is the co-normal direction at the point x on ∂D

(B7) Denote $\alpha(x,t,s) = \int_D \Gamma(x,\xi; t,s)\sigma(\xi,s)d\xi$, where Γ is the

fundamental solution of $D_t U = \mathcal{L}U$. Let $\sigma(\xi,s)$ be uniformly continuous.

Let $\gamma(x,t,s)$ represent either $\alpha(x,t,s)$, $D_i\alpha(x,t,s)$ or $D_i D_j\alpha(x,t,s)$.

Let, uniformly in $\bar{R}$,

$$\int_t^{t'} \gamma^2(x,t',\tau)d\tau + \int_o^t [\gamma(x,t',\tau) - \gamma(x,t,\tau)]^2 d\tau \leq K|t'-t|^\beta \qquad (17)$$

for some real K and $\beta > 0$. Let $D_i D_j D_k \gamma(x,t,s)$ satisfy (17) uniform-

ly for x,t,t' in any compact subset of $D \times [0,T]^2$.

(B8) Let $U(x,0)$ be differentiable w.p.1., and let $a_{ij}(x,0)$ be

continuously differentiable in some neighborhood of ∂D.

Lemma 5. Assume (B1) - (B8). Then there is a random function

$U(x,t)$ which has a version with the following properties, w.p.1. (where

the null set doesn't depend on x,t).

(a) $U(x,t)$ is continuous[†] on $\bar{R}$ (also in quadratic mean);
$(D_i U(x,t))$ is continuous on compact subsets of $\bar{D} \times (0,T]$ (also in

quadratic mean).

(b) The $(D_i D_j U(x,t))$ are continuous on compact subsets of
$D \times (0,T]$.

(c) $U(x,t)$ has an Itô differential which satisfies (16a), for
$t > 0$.

(d) $U(x,t)$ satisfies the boundary condition (16b), and $U(x,t) \to$
$U(x,0)$ as $t \to 0$.

(e) The variances of $U(x,t)$, $(D_i U(x,t))$ (in compact subsets of
$\bar{D} \times (0,T]$) and $(D_i D_j U(x,t))$ (in compact subsets of $D \times (0,T]$) are

---

[†]$D_i U(x,t)$ on ∂D is defined as $\lim_{\substack{y \to x \\ y \in D}} D_i U(x,t)$.

uniformly bounded.

(f) $U(x,t)$ is non-anticipative with respect to the z_t and $\tilde{z}_t$ processes.

Proof. The treatment in Friedman [7], (Theorem 2, p. 144 and Corollary 2, p. 147) will be followed, with the few modifications required by the stochastic nature of the problem taken into account. Define

$$\gamma(x,t) = \int_o^t dz_s \alpha(x,t,s), \quad \gamma_i(x,t) = \int_o^t dz_s D_i \alpha(x,t,s),$$

$$\gamma_{ij}(x,t) = \int_o^t dz_s D_i D_j \alpha(x,t,s), \quad \gamma_{ijk}(x,t) = \int_o^t dz_s D_i D_j D_k \alpha(x,t,s).$$

Let $k(x,t) = \int_o^t dz_s \rho(x,t,s)$. Then

$$Ek^2(x,t) = \int_o^t dt \rho^2(x,t,s)$$

$$Ek^{2n}(x,t) = K_n [Ek^2(x,t)]^n, \text{ for some real } K_n \tag{18}$$

$$E[k(x,t') - k(x,t)]^2 = \int_t^{t'} ds \rho^2(x,t',s) + \int_o^t ds[\rho(x,t',s) - \rho(x,t,s)]^2.$$

Note also that $\gamma_i(x,t)$ is the 'mean square' derivative of $\gamma_o(x,t)$ with respect to the ith coordinate of x in $D \times (0,T]$, and $\gamma_{ijk}(x,t)$ is the 'mean square' derivative of $\gamma_{jk}(x,t)$ with respect to the ith coordinate of x in $D \times (0,T]$.

Then, by the estimates (18), (B7) and Lemma 2, there is a version of $\gamma_o(x,t)$ which (w.p.1.) is continuous on $\bar{R}$; it has continuous derivatives $D_i \gamma_o(x,t) = (D_i \gamma_o(x,t)) = \gamma_i(x,t)$ on $\bar{R}$ and continuous

second derivatives $D_i D_j \gamma_o(x,t) = \gamma_{ij}(x,t) = (D_i D_j \gamma_o(x,t))$ in compact subsets of $D \times [0,T]$. Furthermore, for $(x,t) \in \partial D \times (0,T]$, $\partial/\partial V(x) = \Sigma \varphi_i(x) D_i$, where φ_i are Holder continuous. Hence, the function

$$\frac{\partial}{\partial V(x)} \int_o^t dz_s \alpha(x,t,s) \equiv \gamma_V(x,t)$$

also has a continuous w.p.1. version on $\partial D \times (0,T]$, and in fact, can be identified with $\int_o^t dz_s \frac{\partial \alpha(x,t,s)}{\partial V(x)}$. Next $D_t \alpha(x,t,s) = \mathcal{L}\alpha(x,t,s)$, $s < t$, $x \notin \partial D$, and, by (B7), $\int_o^t (D_t \alpha(x,t,s))^2 ds \leq K < \infty$ on $\bar{R}$. Also $\alpha(x,t,s)$ is continuous on $\bar{R}$ and tends to $\sigma(x,t)$ as $s \uparrow t$. Hence

$$d \int_o^t \alpha(x,t,s) dz_s = \sigma(x,t) dz_t + [\int_o^t \alpha_t(x,t,s) dz_s] dt = \sigma(x,t) dz_t + \mathcal{L} \int_o^t \alpha(x,t,s) dz_s dt.$$

From what has been said the function $F(x,t)$ defined by

$$F(x,t) = \int_D \frac{\partial \Gamma(x,\xi;t,0)}{\partial V(x)} U(\xi,0) d\xi - \int_o^t d\tau \int \frac{\partial \Gamma(x,\xi;t,\tau)}{\partial V(x)} f(\xi,\tau) d\xi$$

$$+ \beta(x,t) \int_D \Gamma(x,\xi;t,0) U(\xi;0) d\xi$$

$$- \beta(x,t) \int_o^t d\tau \int_D \Gamma(x,\xi;t,\tau) f(\xi,\tau) d\tau - \beta(x,t) \gamma_o(x,t) - g(x,t)$$

$$- v(x,t) r(t)$$

is continuous and uniformly bounded w.p.1. on $\partial D \times (0,T]$ (see Friedman [7], p. 145, where continuity is shown for a similar deterministic problem). Then, there is a continuous (and uniformly bounded (w.p.1.) solution on $\partial D \times [0,T]$ to the equation (see Friedman [7], eqn. 3.6, p. 145))

$$\varphi(x,t) = 2 \int_0^t d\tau \int_{\partial D \times [0,T]} [\frac{\partial \Gamma(x,\xi;t,\tau)}{\partial V(x)} + \beta(x,t)\Gamma(x,\xi;t,\tau)]\varphi(\xi,\tau)dS_\xi$$
$$+ 2F(x,t)$$

where dS_ξ is the differential surface measure on ∂D. Finally, (see [7], Theorem 2, p. 144 and Corollary 2, p. 147), it is evident that the function

$$U(x,t) = \int_0^t d\tau \int_{\partial D} \Gamma(x,\xi;t,\tau)\varphi(\xi,\tau)dS_\xi + \int_D \Gamma(x,\xi;t,0)U(\xi,0)d\xi - \gamma_0(x,t)$$
$$- \int_0^t d\tau \int \Gamma(x,\xi;t,\tau)f(\xi,\tau)d\xi$$

has the properties required. In particular, $F(x,t)$ is a non-anticipative functional of the z_t and $\tilde{z}_t$ processes, which implies that $\varphi(x,t)$ and, in turn, $U(x,t)$, are also non-anticipative. Q.E.D.

Now, redefine μ_1 to be the measure determined by $U(x,s)$, $s \le t$, and dy_s given by (B6) for $s \le t$, and $dr(s)$, $s \le t$, given by (B5). Let μ_0 be the measure determined by $U(x,s)$, $r(s)$, $s \le t$, and $w(s)$, $s \le t$.

Let $R(t)$ denote the vector $E_1^t r(t)$, $P_R(t)$ denote the covariance matrix $E_1^t(r(t) - E_1^t r(t))$ $(r(t) - E_1^t r(t))'$ and $P_{MR}(x,t)$ denote the covariance $E_1^t(U(x,t) - E_1^t U(x,t))(r(t) - E_1^t r(t))$.

<u>Theorem 2.</u> <u>Assume</u> (B1) - (B8). <u>Then there is a version of</u> $M(x,t)$ <u>such that</u> w.p.1.: $M(x,t)$ <u>and its first 'mean square' (or true) deriv-</u> <u>ative are continuous w.p.1. on</u> $\bar{R}$ <u>and</u> $\bar{D} \times (0,T]$, resp. <u>The second</u> <u>'mean square' (or true) derivatives of</u> $M(x,t)$ <u>are continuous in</u> $D \times$ $(0,T]$ <u>and</u> $M(x,t)$ <u>has an Itô differential which satisfies</u>

$$dM(x,t) = [\mathscr{L}M(x,t) - f(x,t)]dt + \tag{19a}$$

$$[dy - \int_{\partial D} H(\xi,t)M(\xi,t)dS_\xi]'\Sigma_t^{-1} [\int_{\partial D} H(\xi,t)P(\xi,x,t)dS_\xi].$$

Also

$$\frac{\partial M(x,t)}{\partial V(x)} + \beta(x,t)M(x,t) = g(x,t) + R(x,t) \tag{19b}$$

$$dR(t) = AR(t)dt + [dy - \int_{\partial D} H(\xi,t)M(\xi,t)dS_\xi]'\Sigma_t^{-1} [\int_{\partial D} H(\xi,t)P_{MR}(\xi,t)dS_\xi \tag{20}$$

$$\overset{\bullet}{P}(x,y,t) = (\mathscr{L}_x + \mathscr{L}_y)P(x,y,t) + \sigma(x,t)\sigma(y,t) \tag{21}$$

$$- [\int_{\partial D} H(\xi,t)P(x,\xi,t)dS_\xi]'\Sigma_t^{-1} [\int_{\partial D} H(\xi,t)P(\xi,y,t)dS_\xi]$$

$$\overset{\bullet}{P}_R(t) = A'P_R(t) + P_R(t)A + G(t)G'(t) -$$

$$- [\int_{\partial D} H(\xi,t)P_{MR}(\xi,t)dS_\xi]'\Sigma_t^{-1} [\int_{\partial D} H(\xi,t)P_{MR}(\xi,t)dS_\xi]$$

$$\overset{\bullet}{P}_{MR}(x,t) = \mathscr{L}P_{MR}(x,t) + AP_{MR}(x,t)$$

$$- [\int_{\partial D} H(\xi,t)P(x,\xi,t)dS_\xi]'\Sigma_t^{-1} [\int_{\partial D} H(\xi,t)P_{MR}(\xi,t)dS_\xi]$$

$P(x,y,t)$ satisfies the boundary conditions for (x,t) on $\partial D \times (0,T]$,

$$\frac{\partial P_V(x,y,t)}{\partial V(x)} + \beta(x,t)P(x,y,t) = v(x,t)P_{MR}(x,t)$$

and $P_{MR}(x,t)$ satisfies the (vector) boundary conditions

$$\frac{\partial P_{MR}(x,t)}{\partial V(x)} + \beta(x,t)P_{MR}(x,t) = v(x,t)P_R(t).$$

<u>Proof</u>. The details are very similar to those of Theorem 1 and Lemma 4 and are omitted. Only the boundary conditions will be discussed. By Lemma 5, and a result similar to that of Lemma 4, it is easy to show that there is a version of $M(x,t)$ so that (w.p.1.) $M(x,t)$ is continuous on $\overline{R}$ (and in quadratic mean) $(D_i M(x,t)) = D_i M(x,t)$ is continuous on $\overline{D} \times (0,T]$ (and in quadratic mean). Similarly, for $x \in \partial D$, it can be shown that $\partial M(y,t)/\partial V(x)$ and $\partial U(y,t)/\partial V(x)$ are continuous in quadratic mean on $\overline{D} \times (0,T]$ (as functions of (x,y,t)). Then $E_1^t(\partial U(y,t)/\partial V(x))$ $= \partial M(y,t)/\partial V(x)$, where the last term is defined by

$$\lim_{y \to x, y \in D} \partial M(y,t)/\partial V(x) \equiv \partial M(x,t)/\partial V(x). \quad \text{Also} \quad \lim_{y \to x, y \in D} \partial U(y,t)/\partial V(x)$$

satisfies (w.p.1.)

$$E_1^t[\frac{\partial U(x,t)}{\partial V(x)} + \beta(x,t)U(x,t) - v(x,t)r(t) - g(x,t)]$$

$$= \frac{\partial M(x,t)}{\partial V(x)} + \beta(x,t)M(x,t) - v(x,t)R(t) - g(x,t).$$

The equation

$$E_1^t[U(y,t) - M(y,t)][\frac{\partial U(x,t)}{\partial V(x)} + \beta(x,t)U(x,t) - v(x,t)r(t)$$
$$- g(x,t)] = 0$$

implies

$$\frac{\partial P(x,y,t)}{\partial V(x)} + \beta(x,t)P(x,y,t) - v(x,t)P_{RM}(y,t) = 0$$

Also

$$E_1^t[r(t) - R(t)][\frac{\partial U(x,t)}{\partial V(x)} + \beta(x,t)U(x,t) - v(x,t)r(t) - g(x,t)] = 0$$

implies

$$\frac{\partial P_{MR}(x,t)}{\partial V(x)} + \beta(x,t)P_{MR}(x,t) - v(x,t)P_R(t) = 0.$$

End of details.

REFERENCES

[1]   H. J. Kushner, "An Application of the Sobolev Imbedding Theorems
       to Criteria for the Continuity of Processes with a Vector Para-
       meter, to appear in the Annals of Math. Statist., April, 1969.

[2]   H. J. Kushner, "On the Optimal Control of a System Governed by a
       Linear Parabolic Equation with White Noise Inputs", SIAM J. on
       Control, 6, 1968, pp. 596-614.

[3]   H. J. Kushner, "Dynamical Equations for Nonlinear Filtering",
       Journal of Differential Equations, 3, 1967, pp. 179-190.

[4]   M. Zakai, "On the Optimal Filtering of Diffusion Processes",
       E.E. Publication 80, Faculty of Electrical Engineering, Technion,
       Haifa, Israel, to appear in Z. fur Wahrscheinlichketstheorie.

[5]   M. Loeve, Probability Theory, 3rd Edition, Van Nostrand,
       Princeton, 1963.